*International Journal of Theoretical Physics, Vol. 20, No. 11, 1981*

# Superrenormalizable Quantum Electrodynamics in $n \geqslant 4$

**Jerzy Rayski**

*Institute of Physics, Jagellonian University, Cracow, Poland*

*Received February 5, 1981*

In order to formulate a satisfactory QFT it is not sufficient to secure renormalizability, but rather superrenormalizability. On the other hand, it is not necessary to search for a formalism completely free of infinities. A superrenormalizable QED in $n \geqslant 4$ dimensions may be formulated by introducing a relativistic form factor preserving gauge invariance. This formalism is characterized by a running coupling constant and is asymptotically free. A relation between the bare and the dressed coupling constant is discussed anew.

## 1. A HISTORICAL INTRODUCTION

In 1948 Peierls and McManus (1948) formulated a nonlocal electrodynamics involving a form factor $F(x)$ in the Lagrangian of interaction

$$\mathfrak{L}'(x)=A_\mu(x)\tilde{j}^\mu(x) \tag{1}$$

where

$$\tilde{j}^\mu(x)=\int dx'\,F(x-x')\,j^\mu(x') \tag{2}$$

Hereby $dx$ means a four-dimensional volume element, and the integration is extended over the *whole* space-time (unless it is stated explicitly otherwise). We may call $j^\mu$ and $\tilde{j}^\mu$ a "true" or an "effective" charge and current density, respectively. The Fourier transform of the form factor may depend on $k$ only via the square $k^2=k_0^2-\mathbf{k}^2$

$$\frac{1}{(2\pi)^4}\int dx F(x)e^{-ikx}=f(k^2/M^2) \tag{3}$$



0020-7748/81/1100-0881$03.00/0 

where $M$ is a constant with dimension of mass (in natural units $c=\hbar=1$). For correspondence reasons it is necessary to assume

$$f(0)=1 \tag{4}$$

so that in the limit $M\to\infty$ the form factor becomes a (four-dimensional) Dirac delta function, and the theory goes over into the usual, local electrodynamics.

The above modification of electrodynamics spoils neither covariance under the Poincaré group nor gauge invariance. In view of the invariance under space and time translations there should exist local conservation laws for momentum and energy. The locally conserved quantities were found independently by Rzewuski (1953) and Pauli (1953).

The transformation

$$A_\mu \to A_\mu + \partial_\mu\Theta/e \tag{5}$$

supplemented by the phase transformation[1]

$$\psi \to e^{-i\tilde{\Theta}}\psi \tag{6}$$

where

$$\tilde{\Theta}(x)=\int dx'\Theta(x')F(x'-x) \tag{7}$$

leaves the formalism gauge invariant, provided the integration in (7) is exchangeable with partial differentiations

$$\partial_\mu\tilde{\Theta}=\widetilde{\partial_\mu\Theta} \tag{8}$$

This constitutes a condition upon the class of functions $\Theta$. It is sufficient to assume that $\Theta$ decreases quickly at infinity in order to secure (8). Thus, the theory is invariant under a slightly restricted gauge group.

If the Fourier transform (3) of the form factor decreases to zero for $|k^2|\to\infty$, the convergence situation may be improved. Nevertheless, this QED was abandoned for the following reasons: (i) It presented serious difficulties at quantization (Marnelius, 1972), chiefly in connection with the problem of chronological ordering of nonlocal interaction terms. (ii) It was unable to avoid all the convergence difficulties; in particular it was of no

[1] We assume a Dirac spinor field to constitute the sources of the electromagnetic field.

help as regards the problem of an infinite vacuum polarization.[2] Also higher-order corrections to the electron self-energy remained infinite (while the concept of superrenormalizability was unknown yet). Last but not least, this theory was accused of violating not only microcausality, but even macrocausality inasmuch as a relativistically invariant form factor smears out the interaction uniformly along the hyperboloids $(x-x')^2=\text{const}$.

The above-mentioned difficulties are not decisive. It is possible to formulate a quantum theory corresponding closely to the above classical nonlocal QED although the mere concept of field equations will lose its sense except for free fields. This new theory will be shown to be superrenormalizable and satisfying the requirements of causality.

## 2. A SEMIAUTONOMOUS $S$ MATRIX

Let us consider the action integral

$$W=W^{(0)}+W' \tag{9}$$

where $W^{(0)}$ is the usual action integral for the free fields $A_\mu$ and the spinor field $\psi$

$$W^{(0)}=\int dx\,\mathfrak{L}^{(0)} \tag{10}$$

whereas the interaction term $W'$ will be assumed in a slightly more general form in comparison with the form of Peierls et al.

$$W'=\int dx\,\tilde{A}_\mu \tilde{j}^\mu \tag{11}$$

where

$$\tilde{A}_\mu(x)=\int dx\,F(x-x')A_\mu(x') \quad \text{and} \quad \tilde{j}^\mu=\int dx\,G(x-x')j^\mu(x') \tag{12}$$

where $F$ and $G$ are two form factors.

[2] This could be remedied by a more general type of form factor smearing $A_\mu$, $\psi$, and $\bar{\psi}$ independently. This, however, spoils the gauge invariance.

In spite of the fact that (11) looks as if it represented an integral whose integrand is a Lagrangian of interaction

$$\mathfrak{L}'(x)=\tilde{A}_\mu(x)\tilde{j}^\mu(x) \tag{13}$$

the above theory is *not* of the usual Lagrangian and Hamiltonian character due to the circumstance that the fields $A_\mu$ and $\psi$ involved in (13) are taken at different instants of time. If it were of the usual character then we could quantize it according to the usual rules and derive the operator

$$T\exp i\int dx\,\mathfrak{L}'(x)$$

where $T$ denotes an operator of chronological ordering of products $L'(x)L'(y)\ldots$ according to whether $x^0$ is earlier or later than $y^0$. However, in consequence of the nonlocal character of the interaction the Lagrangians at different points $L'(x), L'(y),\ldots$ do not commute even for spacelike distances between $x$ and $y$, and consequently, the above operator violates the relativistic requirements. Thus, the nonlocal theory is not quantizable in the usual way.

One could object that this inconsistency applies only to the formulation in the interaction picture whereas a formulation in the Heisenberg picture may exist and be correct. However, as was shown by Marnelius (1972) and others, it is also not the case. A quantization in the Heisenberg picture using the Yang–Feldman (1950) formalism cannot be performed in a consistent way either.

The way out is possible, but requires a bold assumption: we have to give up the usual requirements that quantum operators representing the fields $A_\mu$ and $\psi$ have to satisfy field equations of the same form as their classical counterparts do, but *define* an $S$ matrix (or an operator of evolution in time) in an axiomatic way. The only criteria for the autonomous $S$ operator are those of existence (finiteness of the matrix elements), unitarity, causality, Lorentz covariance, and correspondence.

Our definition of the $S$ operator is

$$S=\underset{(a)}{T}\;\underset{(j)}{T}\exp ie\int dx\,\tilde{A}_\mu\tilde{j}^\mu \tag{14}$$

with the fields $A_\mu$ and $\psi$ satisfying the free field equations and the usual commutation relations valid for free fields, in particular

$$\left[A_\mu(x), j^\nu(y)\right]=0 \tag{15}$$

for arbitrary 4-points $x$ and $y$ (which is not the case for the smeared-out fields $\tilde{A}_\mu$ and $\tilde{j}^\nu$). The symbols $T_{(A)}$ and $T_{(j)}$ involved in (14) denote two independent operators of chronological ordering of $A_\mu$ (but not $\tilde{A}_\mu$) and $j^\nu$ (but not $\tilde{j}^\nu$) according to the sequence of their respective arguments. Such operators of chronological orderings possess a covariant meaning inasmuch as the respective field operators subjected to chronological ordering all commute for spacelike distances.

Thus, the new $S$ operator (14) is a relativistically covariant concept. Obviously it satisfies also the requirement of correspondence: First of all, it goes over into the usual Dyson's operator in the local limit when both $F$ and $G$ become Dirac delta functions. Moreover, it also goes over into the usual operator if one of the fields, either the electromagnetic field or the charged field, is an external, classical field. In view of a close correspondence of (14) with the traditional $S$ operator derivable from a Lagrangian and Hamiltonian electrodynamics, it should be called a semiautonomous rather than autonomous $S$ operator.

It should be stressed that (14) is a *definition* of the $S$ operator. Such an operator is not derivable from some more primary principles (e.g., from a Schrödinger equation or from Heisenberg's equations of motion). If the definition (14) is assumed to be a fundamental, primary assumption, then the field equations for the interacting field operators in the Heisenberg picture do not exist at all. There exist only their classical analogs.

## 3. THE OPERATOR OF EVOLUTION IN TIME

It is possible and plausible to define—by means of a straightforward interpolation—an operator of evolution in finite time intervals, or between two spacelike hypersurfaces of measurements $\sigma_1$ and $\sigma_2$. This operator is

$$S_{(1)}^{(2)} = \underset{(A)}{T}\,\underset{(j)}{T} \exp ie \int_{\sigma_1}^{\sigma_2} dx\, \tilde{A}_\mu(x) \tilde{j}^\mu(x) \tag{16}$$

In this way we do not need to assume that only measurements of asymptotic states make sense, but also measurements at an arbitrary time instant and states attached to arbitrary spacelike hypersurfaces $\sigma$ are meaningful. Inasmuch as only a single integration is extended between finite limits (whereas the integrations involved in the "smearing out" of the fields are always extended from $-\infty$ to $+\infty$), the operator (16) is multiplicative:

$$S_{(1)}^{(3)} = S_{(1)}^{(2)} \cdot S_{(2)}^{(3)} \tag{17}$$

which shows that the transformations of the state in the course of time form a group. Inasmuch as the generator of the infinitesimal transformations

$$L=e\int d^3x\tilde{A}_\mu(x)\tilde{j}^\mu(x) \tag{18}$$

is Hermitian, these transformations are unitary.

This was rather a formal argument for unitarity. Inasmuch as in theories with an infinite number of degrees of freedom it often happens that a property which is formally satisfied is violated in practice (e.g., the appearance of a nonvanishing photon self-mass in spite of a formal gauge invariance) the same may happen with unitarity. But even this mischief would not kill the nonlocal QED because a probabilistic interpretation may be easily restored by means of a renormalization of probabilities:

$$P_{i\to f}=\frac{1}{N(i)}\langle i|S^\dagger|f\rangle\langle f|S|i\rangle \tag{19}$$

where $p_{i\to f}$ means probability for a transition from $|i\rangle$ at $\sigma_1$ to $|f\rangle$ at $\sigma_2$ while the normalizing factor is

$$N(i)=\langle i|S^\dagger S|i\rangle \tag{19'}$$

This redefinition may be also stated briefly as follows: only direction but not length of vectors in Hilbert space is physically meaningful.

To be stressed once more: The operator of evolution in time (16) plays the same fundamental role in our theory as, e.g., the time-dependent Schrödinger equation (and Dyson's operator derivable from it) does in the usual local theory. The difference is that now the operator (16) is primary and not derivable from anything else.

## 4. THE GRAPHS AND THE FEYNMAN RULES

The above formalism leads to the following rules: the usual prescriptions, graphs, and Feynman rules well known from the local QED hold true except for a modification of vertices consisting in a replacement of the coupling constant $e$ by

$$e\to ef(k)g(k) \tag{20}$$

where $f$ and $g$ mean Fourier transforms of the form factors $F$ and $G$ while $k$ means the four-momentum of the photon line issuing from or running into the vertex in question.

In the case of finite limits of integration ($\sigma_1$ and $\sigma_2$) we have to take $g(p-q)$ instead of $g(k)$ where $p$ and $q$ are the arguments of $\bar{\psi}$ and $\psi$ while an integration of $\exp i(p-q\pm k)x$ between finite limits is still left to be performed.

## 5. CAUSALITY

In order to secure causality we assume $F$ and $G$ to be of the form of retarded or causal Green's functions (fundamental solutions) of a Klein–Gordon equation with a very large mass. Their Fourier transforms are

$$f(k)=\frac{M_1^2}{k^2+M_1^2} \quad \text{and} \quad g(k)=\frac{M_2^2}{k^2+M_2^2} \tag{21}$$

with a suitable prescription of bypassing the poles.

The introduction of causal or retarded form factors may be objected since it violates invariance under time reversal. There are two alternatives: (i) one may refute the objection by pointing out that one direction along the time axis is indeed privileged in nature, so that—sooner or later—one shall have to account for this asymmetry also on the level of fundamental (i.e., dynamical) physical laws. (ii) It is also possible to restore a full symmetry by assuming that the operator (14) or (16) is applicable only for probabilistic predictions of the future, whereas for possible retrodictions (of an unknown past) one should use another $S$ operator with the retarded (or causal) form factors replaced by the advanced (or anticausal) ones.

## 6. SUPERRENORMALIZABILITY

In view of the prescription (20) with (21) it is seen that each propagator visualized by an internal photon line acquires a factor $(k^2)^{-5}$ instead of the usual $(k^2)^{-1}$ for $|k^2|\to\infty$. This removes the divergences of the self-energy type for the charged particles, but it does not help in the least in the problem of the vacuum polarization as visualized by the particle–antiparticle closed loop. In fact, this formalism affects only vertices and consequently the photon lines, but not the propagator of the electron. However, it is easily seen that the higher-order corrections to the vacuum polarization yield already finite results. A simple power counting ascertains one that this version of QED is superrenormalizable. In superrenormalizable theories there are still some infinities left, but they may be removed by means of a finite number of counterterms being polynomials (but not infinite series) in the coupling constant. As was shown by Glimm and Jaffe (1968) and others, only superrenormalizable theories may be formulated successfully in a rigorous and mathematically satisfactory way.

The traditional models of interactions become superrenormalizable if considered in a space-time with a diminished number of dimensions ($n=2$ or $n=3$). Our version based on (20) with (21) is superrenormalizable also in a six-dimensional space-time. This may be of importance for the so-called "grand unification" necessitating a number of additional (compact) dimensions of space-time.

## 7. A RELATION BETWEEN THE BARE AND DRESSED COUPLING CONSTANT

Inasmuch as the above nonlocal theory is superrenormalizable it is sufficient to perform a charge renormalization in the lowest order of approximation, i.e., to ensure a finite contribution from the simplest vacuum polarization graph (one loop) to ensure finiteness of all higher-order contributions from arbitrarily complicated graphs.

Assuming the charged particle vacuum, but a presence of an external current $J_\mu$ we get the following equation:

$$\Box A_\mu = -e_0 J_\mu - e_0 \langle 0 | j_\mu | 0 \rangle \tag{22}$$

where $e_0$ means a bare charge and $j_\mu$ is the charge and current operator of the quantized fields.

In the lowest order of the perturbation calculus one gets

$$\langle 0 | j_\mu | 0 \rangle = -e_0 \left( b_0 - b_1 \Box + b_2 \Box^2 + \cdots \right) A_\mu \tag{23}$$

where for the case of a spin-1/2 field

$$b_0 = \frac{m^2}{3\pi^2} \quad \text{and} \quad b_1 = \frac{1}{18\pi^2} + \frac{1}{24\pi^2} \int_{-\infty}^{\infty} \frac{d\alpha}{\alpha} e^{im^2\alpha} \operatorname{sgn} \alpha \tag{24}$$

It should be noticed that all coefficients $b_n$ except for $b_1$ are finite. Obviously, the meaningless term with $b_1$ should be absorbed by a charge renormalization $e_0 \to e$ (whereas $b_0$ is to be removed by a mass renormalization). From (22) and (23) it is seen that the renormalized coupling constant is

$$e = \frac{e_0}{1 + b_1 e_0^2} \tag{25}$$

The above formula holds true in the lowest order of the perturbation calculus, but inasmuch as higher-order corrections are finite, this formula is already suitable for a discussion of the problems arising from the appearance of infinite integrals.

Using some estimates more or less equivalent to the formula (25) several authors (Landau and Pomeranchuk, 1955) claimed that the renormalized coupling constant $e$ must be exactly zero and consequently the usual local QED breaks down completely. On the other hand, an electrodynamics with a cutoff, i.e., with a finite $b_1$, would be characterized by $0<e<e_0$ (unless the cutoff is unreasonably large, in which case $b_1$ may become negative). The pretended inequality $e<e_0$ was explained as follows: "the bare charge is screened by a surrounding cloud of virtual particles with opposite charge whereas that with equal sign of the charge is removed to infinity."

In our opinion such conclusions ($e_0>e$ and $e\to 0$ if the cutoff is removed) are erroneous, whereas a correct reasoning runs as follows: Starting with a cutoff (i.e., with a finite coefficient $\beta$ instead of the infinite $b_1$) the equation (25) may be solved with respect to $e_0$ while $e$ is assumed to be finite and known (close to the square root of $1/137$). From the two roots

$$e_0=\left[1\pm(1-4e^2\beta)^{1/2}\right]/2e\beta \tag{26}$$

one has to choose the root with the minus sign since in the limit $\beta\to 0$ we must have $e_0\to e$. From this it is seen that by introducing a movable cutoff (representing a variable smearing out of the charge) we encounter an effect similar to the transitions of the phase (of condensation of the charged matter). For $\beta<1/4e^2$ we have one phase, but increasing $\beta$ beyond $1/4e^2$ the bare coupling constant becomes complex

$$e_0=\left[1-i(4e^2\beta-1)^{1/2}\right]/2e\beta \tag{27}$$

which means a change of the phase. The unexpected feature of a complex $e_0$ disappears in the limit $\beta\to\infty$, i.e., when removing the cutoff. Indeed, with increasing $\beta$ the bare coupling constant tends (like $1/i\beta^{1/2}$) to the limit $e_0\to 0$ from the domain of imaginary values whereas $e$ remains always finite and real, exactly as in the Lee model.

Inasmuch as after the removal of the cutoff the bare coupling constant becomes zero, there are no more problems with unitarity and the whole situation becomes intelligible: In order to avoid inconsistencies one has to assume a *vanishing bare coupling constant*! It is just the effect of dressing that produces a *finite* effective coupling from a vanishing bare coupling.

However, the limit transition $e_0 \to 0$ together with the removal of the cutoff is a subtle procedure, otherwise we could fall into inconsistencies (either $e=0$ or $e=\infty$). It is clear that by proceeding not carefully enough one can easily run into contradictions, e.g., encounter "ghosts," violate unitarity, or destroy completely the interaction and obtain a unit $S$ matrix.

The above analysis does not apply merely to the nonlocal theory but also to the usual QED because the form factor (3) or (21) may be regarded as another cutoff which has to be removed by the limit transition $M \to \infty$ at the very end, after the charge and mass renormalizations.

## 8. OUTLOOK

A mistake in the early stage of development of quantum field theory was that we were too ambitious: we wanted to remove all infinities at once, by means of a single paradigm. It explains why the electrodynamics of Peierls was rejected or forgotten. Today we see that such ambitions were false. As far as ultraviolet divergences are concerned it is sufficient to secure a superrenormalizability, and this has been granted by the above-described nonlocal modification of QED.

Another paradigm—certainly not competitive, but rather supplementing, the above—consists in introducing a suitable mixture of compensating fields which also converts the QED into a superrenormalizable form. We shall discuss it elsewhere.

The possibilities of extending the above formalism to the case of other gauge fields exist, are promising, and may explain both asymptotic freedom and confinement.

Let us finally mention the fact that the possibility of introducing a form factor of the type of a causal or retarded function for a Klein–Gordon equation with mass $M$ means a departure from an absolute invariance of dynamical theories under time reversal. This may be of some interest because a privileged direction along the time axis will have to be explained —sooner or later—on the level of dynamical theories.

## REFERENCES

Glimm, J., and Jaffe, A. (1968). *Physical Review*, **176**, 1945.
Landau, L., and Pomeranchuk (1955). *Doklady Akademii Nauk SSSR*, **102**, 489.
Marnelius, R. (1972). *Physical Review D*, **8**, 2472.
McManus, H. (1948). *Proceedings of the Royal Society of London, Series A*, **195**, 322.
Pauli, W. (1953). *Nuovo Cimento*, **10**, 648.
Rzewuski, J. (1953). *Nuovo Cimento*, **10**, 182.
Yang, C. N., and Feldman, D. (1950). *Physical Review*, **79**, 972.